澳門大學 UNIVERSIDADE DE MACAU UNIVERSITY OF MACAU

澳門理工學院 Instituto Politécnico de Macau Macao Polytechnic Institute

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澳門科技大學 UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2021 年試題及參考答案
2021 Examination Paper and Suggested Answer

數學附加卷 Mathematics Supplementary Paper

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共 22 版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。

3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。

4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。

5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。

6. 本考卷的圖形並非按比例繪畫。

7. 考試中不可使用任何形式的計算機。

8. 請用藍色或黑色原子筆作答。

9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page – 22 pages

   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.

3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.

4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.

5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.

6. The diagrams in this examination paper are not drawn to scale.

7. Calculators of any kind are not allowed in the examination.

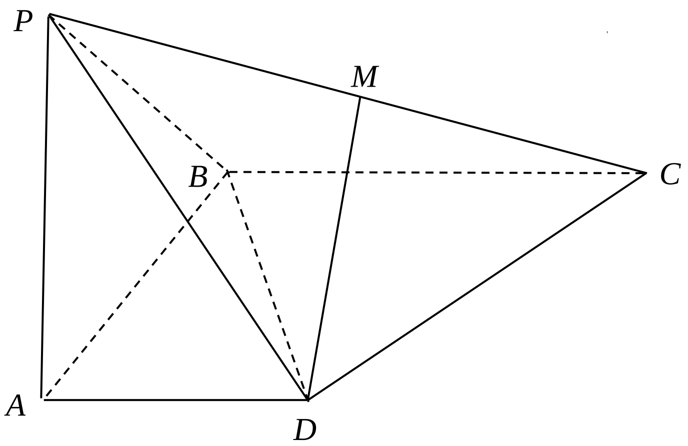
8. Answer the questions with a blue or black ball pen.

9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1.



如上圖所示，*ABCD* 為梯形，*PA* 垂直 *ABCD*，$\angle DAB = \angle ABC = \frac{\pi}{2}$，$|AD| = 1$，

$|PA| = |AB| = |BC| = 2$。設 *M* 為線段 *PC* 的中點。

(a) (i) 求三角形 PBD 的面積。 (6 分)

(ii) 求三棱錐 *P-ABD* 的體積，從而求點 *A* 至平面 PBD 的距離。 (4 分)

(b) 證明

(i) *CB* 垂直平面 PAB。 (3 分)

(ii) *DM* 與平面 PAB 平行。

[提示: 設 *N* 為 *PB* 的中點。證明 *ADMN* 是一長方形。] (7 分)

As shown in the above figure, *ABCD* is a trapezoid, *PA* is perpendicular to *ABCD*, $\angle DAB = \angle ABC = \frac{\pi}{2}$, $|AD| = 1$, $|PA| = |AB| = |BC| = 2$. Let *M* be the midpoint of *PC*.

(a) (i) Find the area of the triangle PBD. (6 marks)

(ii) Find the volume of the triangular pyramid *P-ABD*, and hence find the distance from point *A* to the plane PBD. (4 marks)

(b) Show that

(i) *CB* is perpendicular to the plane PAB; (3 marks)

(ii) *DM* is parallel to the plane PAB;

[Hint: Let *N* be the midpoint of *PB*. Show that *ADMN* is a rectangle.] (7 marks)

2. (a) 設 $f(x) = 2x^3 - 9x^2 + 12x - 5$。

(i) 求 $f'(x)$ 及 $f''(x)$。 (2 分)

(ii) 求 $f(x)$ 的局部極大值和局部極小值。 (4 分)

(iii) 求曲線 $y = f(x)$ 的拐點。 (2 分)

(iv) 繪出曲線 $y = f(x)$，$-1 \leq x \leq 3$。 (3 分)

(v) 繪出曲線 $y = f(|x|) - 1$，$-1 \leq x \leq 3$。 (1 分)

(b) 求由曲線 $y = -x^2 + 3x$ 及曲線 $y = 2x^3 - x^2 - 5x$ 所包圍的區域的面積。 (8 分)

(a) Let $f(x) = 2x^3 - 9x^2 + 12x - 5$.

(i) Find $f'(x)$ and $f''(x)$. (2 marks)

(ii) Find the local maximum and local minimum values of $f(x)$. (4 marks)

(iii) Find the inflection point of the curve $y = f(x)$. (2 marks)

(iv) Sketch the curve $y = f(x), -1 \leq x \leq 3$. (3 marks)

(v) Sketch the curve $y = f(|x|) - 1, -1 \leq x \leq 3$. (1 marks)

(b) Find the area of the region bounded by the curves $y = -x^2 + 3x$ and $y = 2x^3 - x^2 - 5x$. (8 marks)

3. 已知拋物線 $P: x^2 = 8y$。

(a) 若直線 $L: y = mx + c$ 與拋物線 P 相切，證明 $c = -2m^2$。 (4 分)

(b) 設 L_1 及 L_2 為拋物線 P 的兩條不同的切線，其斜率分別為 m_1 及 m_2，且相交於點 $A(h,k)$。

(i) 求點 A，答案以 m_1 及 m_2 表示。 (6 分)

(ii) 若 L_1 及 L_2 的夾角為 $\frac{\pi}{2}$，求點 A 的軌跡。 (5 分)

(iii) 若 L_1 及 L_2 的夾角為 $\frac{\pi}{4}$ 及 $m_1 = 2$，求點 A。 (5 分)

Given parabola $P: x^2 = 8y$.

(a) If the straight line $L: y = mx + c$ is tangent to the parabola P, show that $c = -2m^2$. (4 marks)

(b) Suppose L_1 and L_2 are two different tangent lines of P with slopes m_1 and m_2, respectively, and intersect at point $A(h,k)$.

(i) Find the point A, giving your answer in terms of m_1 and m_2. (6 marks)

(ii) If the angle between L_1 and L_2 is $\frac{\pi}{2}$, find the locus of point A. (5 marks)

(iii) If the angle between L_1 and L_2 is $\frac{\pi}{4}$ and $m_1 = 2$, find the point A. (5 marks)

4. (a) 已知 恆等式 $\sin(X+Y) = \sin X \cos Y + \cos X \sin Y$，證明恆等式

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}。\qquad (2 分)$$

(b) 設 $A+B+C=\pi$，證明 $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$。 (7 分)

(c) (i) 用數學歸納法，證明對任意正整數 n，

$$2\sin x\,[\cos x + \cos 3x + \cdots + \cos(2n-1)x] = \sin 2nx。\qquad (7 分)$$

(ii) 用 (i) 的結果，解 $\cos x + \cos 3x + \cos 5x = 0$ ，其中 $0 \le x \le 2\pi$。 (4 分)

(a) Given the identity $\sin(X+Y) = \sin X \cos Y + \cos X \sin Y$. Prove the identity

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}. \qquad (2 \text{ marks})$$

(b) Suppose $A+B+C=\pi$. Show that $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$. (7 marks)

(c) (i) Use mathematical induction to show that for any positive integer n,

$$2\sin x\,[\cos x + \cos 3x + \cdots + \cos(2n-1)x] = \sin 2nx. \qquad (7 \text{ marks})$$

(ii) Using the result in (i), solve $\cos x + \cos 3x + \cos 5x = 0$, where $0 \le x \le 2\pi$. (4 marks)

5. (a) 因式分解行列式 $\begin{vmatrix} a & a^2+1 & bc \\ b & b^2+1 & ac \\ c & c^2+1 & ab \end{vmatrix}$。 (8 分)

(b) 已知以 x、y 和 z 為未知量的方程組:

$$(E):\begin{cases} kx + \ \ y - z \ = p \\ \ \ x + ky + \ z \ = q \\ -x + \ \ y + kz = r \end{cases},$$

其中 k, p, q, r 為常數。

(i) 求 k 的取值範圍，使得 (E) 有唯一解。 (4 分)

(ii) 設 $k = 2$，且 (E) 有多於一個解。求 p, q, r 的關係，並於 $p = 5, q = 1, r = -4$ 時，解方程組 (E)。 (8 分)

(a) Factorize the determinant $\begin{vmatrix} a & a^2+1 & bc \\ b & b^2+1 & ac \\ c & c^2+1 & ab \end{vmatrix}$. (8 marks)

(b) Given the system of equations with unknowns x, y and z:

$$(E):\begin{cases} kx + \ \ y - z \ = p \\ \ \ x + ky + \ z \ = q \\ -x + \ \ y + kz = r \end{cases},$$

where k, p, q, r are constants.

(i) Find the range of k such that (E) has a unique solution. (4 marks)

(ii) Suppose $k = 2$ and (E) has more than one solution. Find the relation between p, q, and r, and solve the system (E) when $p = 5$, $q = 1$, $r = -4$. (8 marks)

參考答案：

1. (a) (i) 計算得知 $|PD|=\sqrt{|PA|^2+|AD|^2}=\sqrt{5}$， $|BD|=\sqrt{|BA|^2+|AD|^2}=\sqrt{5}$ 及

$|PB|=\sqrt{|PA|^2+|AB|^2}=\sqrt{8}$，故 ΔPBD 是一等腰三角形。若以 PB 為底，

其高為 $\sqrt{|PD|^2-(\frac{1}{2}|PB|)^2}=\sqrt{3}$。因此，$\Delta PBD$ 的面積為 $\sqrt{6}$。

(ii) $P-ABD$ 的體積是 $\frac{1}{3}|PA|(\Delta ABD \text{ 的面積})=\frac{2}{3}$。

設 d 為點 A 至平面 PBD 的距離，則 $A-PBD$ 的體積是 $\frac{d\sqrt{6}}{3}$。

由 $\frac{d\sqrt{6}}{3}=\frac{2}{3}$，得 $d=\sqrt{\frac{2}{3}}$。

(b) (i) 從 $PA\perp ABCD$ 得 $DA\perp AP$，連同 $DA\perp AB$，故 $AD\perp PAB$。

因 AD 與 BC 平行，故 $BC\perp PAB$。

(ii) 從 (b)(i) 的證明，知 $AD\perp PAB$，故 $AD\perp AN$。…..(1)

因 N 和 M 分別為 PB 和PC的中點，故 $NM\parallel BC$。連同 $AD\parallel BC$，

得 $NM\parallel AD$。因此，點 A、D、M 和 N 是共面的;並且由 (1) 知

$NM\perp AN$ …..(2)

因 N 和 M 分別為 PB 和 PC 的中點，故 $|MN|=\frac{1}{2}|BC|=1=|AD|$。…..(3)

綜合以上 (1) - (3)，$ADMN$ 是一長方形。因此，$DM\parallel AN$，

故 DM 與平面 PAB 平行。

2. (a) (i) $f'(x)=6x^2-18x+12$，$f''(x)=12x-18$。

(ii) $f'(x)=0\Leftrightarrow x=1$ 或 $x=2$。

當 $x<1$ 時，$f'(x)>0$，故 $f(x)$ 是遞增的。

當 $1<x<2$ 時，$f'(x)<0$， 故 $f(x)$ 是遞減的。

當 $2<x$ 時，$f'(x)>0$， 故 $f(x)$ 是遞增的。

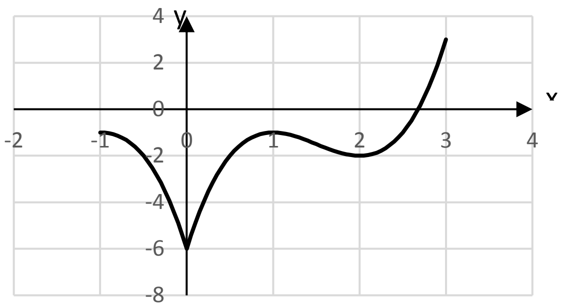
因此，$f(1)=0$ 是一局部極大值，$f(2)=-1$ 是一局部極小值。

(iii) $f''(x)=0\Leftrightarrow x=\frac{3}{2}$。當 $x<\frac{3}{2}$ 時，$f''(x)<0$；當 $\frac{3}{2}<x$ 時，$f''(x)>0$。

因此，曲線 $y=f(x)$ 的拐點是 $(\frac{3}{2},-\frac{1}{2})$。

(iv)

(v)



(b) 解 $\begin{cases} y = -x^2 + 3x \\ y = 2x^3 - x^2 - 5x \end{cases}$，得 $x = -2$ 或 $x = 0$ 或 $x = 2$。

當 $-2 < x < 0$，曲線 $y = -x^2 + 3x$ 是在曲線 $y = 2x^3 - x^2 - 5x$ 之下。

當 $0 < x < 2$，曲線 $y = -x^2 + 3x$ 是在曲線 $y = 2x^3 - x^2 - 5x$ 之上。

故所求面積為

$$\int_{-2}^{0} 2x^3 - x^2 - 5x - (-x^2 + 3x)\,dx + \int_0^2 -x^2 + 3x - (2x^3 - x^2 - 5x)dx$$

$$= \int_{-2}^{0} 2x^3 - 8x\,dx + \int_0^2 8x - 2x^3\,dx$$

$$= \left[\frac{x^4}{2} - 4x^2\right]_{-2}^{0} + \left[4x^2 - \frac{x^4}{2}\right]_0^2$$

$$= 8 + 8$$

$$= 16$$。

3. (a) (i) 由 $\begin{cases} y = mx + c \\ x^2 = 8y \end{cases}$，得 $x^2 - 8mx - 8c = 0$。…… (1)

當 L 與 P 相切，(1)有重根，其判別式為 0，從而得 $c = -2m^2$。

(b) (i) 由 (a)，設 L_i 的方程為 $y_i = m_i x - 2m_i^2$， $i = 1, 2$。因 $A(h, k)$ 為 L_1 及 L_2 的交點，有 $\begin{cases} m_1 h - k = 2m_1^2 \\ m_2 h - k = 2m_2^2 \end{cases}$，從而得 $(h, k) = (2(m_1 + m_2), 2m_1 m_2)$。

(ii) 因 L_1 與 L_2 垂直，得 $m_1m_2=-1$，故 $k=2m_1m_2=-2$。

從 $m_2=-\frac{1}{m_1}$，得 $h=2(m_1-\frac{1}{m_1})$ 。

對任意實數 h ，考慮以 m 為未知量的方程 $2m^2-hm-2=0$，因其判別式 $h^2+16>0$，故知此方程有實數解。由此，知 $h=2(m_1-\frac{1}{m_1})$ 可達到任意實值。因此，點 A 的軌跡是直線 $y=-2$。

(iii) 因 L_1 及 L_2 的夾角為 $\frac{\pi}{4}$ ，故 $1=\tan\frac{\pi}{4}=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$。

當 $m_1=2$，有 $\left|\frac{2-m_2}{1+2m_2}\right|=1$，即 $3m_2^2+8m_2-3=0$。其解為 $m_2=-3$ 或 $m_2=\frac{1}{3}$，故點 A 是 $(-2,-12)$ 或 $(\frac{14}{3},\frac{4}{3})$ 。

4. (a) 由 $\sin A=\sin(\frac{A+B}{2}+\frac{A-B}{2})=\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

及

$$\sin B=\sin(\frac{B+A}{2}+\frac{B-A}{2})=\sin\frac{B+A}{2}\cos\frac{B-A}{2}+\cos\frac{B+A}{2}\sin\frac{B-A}{2}$$
$$=\sin\frac{A+B}{2}\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\sin\frac{A-B}{2}\ ,$$

得 $\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$。

(b)

$$\begin{aligned}
&\sin A+\sin B+\sin C\\
&=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin C\\
&=2\sin\frac{\pi-C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}\\
&=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}\\
&=2\cos\frac{C}{2}(\cos\frac{A-B}{2}+\sin\frac{C}{2})\\
&=2\cos\frac{C}{2}(\cos\frac{A-B}{2}+\sin\frac{\pi-(A+B)}{2})\\
&=2\cos\frac{C}{2}(\cos\frac{A-B}{2}+\cos\frac{A+B}{2})\\
&=2\cos\frac{C}{2}(\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+(\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}))\\
&=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}
\end{aligned}$$

(c)(i) 設 $P(n)$ 代表命題 " $2\sin x\,[\cos x+\cos 3x+\cdots+\cos(2n-1)x]=\sin 2nx$ "。

當 $n=1$，因 $2\sin x\cos x=\sin 2x$，$P(1)$ 成立。

假設對某正整數 k，$P(k)$ 成立。當 $n=k+1$，

$$
\begin{aligned}
&2\sin x\,[\cos x+\cos 3x+\cdots+\cos(2(k+1)-1)x]\\
&=\sin 2kx+2\sin x\cos(2k+1)x\\
&=\sin 2kx+2\sin\tfrac{(2k+2)x-2kx}{2}\cos\tfrac{(2k+2)x+2kx}{2}\\
&=\sin 2kx+\sin(2k+2)x+\sin(-2kx)\\
&=\sin 2(k+1)x，
\end{aligned}
$$

故 $P(k+1)$ 也成立。

根據數學歸納法原理，$P(n)$ 對任意正整數 n 都成立。

(ii) 明顯，當 $x=0,\pi,2\pi$，$\sin x=0$ 及 $\cos x+\cos 3x+\cos 5x\neq 0$。

由 (i)，當 $0\leq x\leq 2\pi$，$\cos x+\cos 3x+\cos 5x=0$ 當且僅當 $\sin 6x=0$

及 $x\neq 0,\pi,2\pi$。解 $\sin 6x=0$，得 $x=\frac{n\pi}{6}$，$n=1,2,3,4,5,7,8,9,10,11$。

5. (a)

$$
\begin{aligned}
\begin{vmatrix} a & a^2+1 & bc\\ b & b^2+1 & ac\\ c & c^2+1 & ab\end{vmatrix}
&=\begin{vmatrix} a & a^2+1 & bc\\ b-a & b^2-a^2 & c(a-b)\\ c-a & c^2-a^2 & b(a-c)\end{vmatrix}\\
&=(b-a)(c-a)\begin{vmatrix} a & a^2+1 & bc\\ 1 & b+a & -c\\ 1 & c+a & -b\end{vmatrix}\\
&=(b-a)(c-a)\begin{vmatrix} a & a^2+1 & bc\\ 1 & b+a & -c\\ 0 & c-b & c-b\end{vmatrix}\\
&=(b-a)(c-a)(c-b)\begin{vmatrix} a & a^2+1 & bc\\ 1 & b+a & -c\\ 0 & 1 & 1\end{vmatrix}\\
&=(b-a)(c-a)(c-b)\begin{vmatrix} a & a^2+1-bc & bc\\ 1 & b+a+c & -c\\ 0 & 0 & 1\end{vmatrix}\\
&=(b-a)(c-a)(c-b)[a(a+b+c)-(a^2+1-bc)]\\
&=(a-b)(b-c)(c-a)(ab+bc+ac-1)
\end{aligned}
$$

(b) (i) (E) 有唯一解的條件是 $\begin{vmatrix} k & 1 & -1\\ 1 & k & 1\\ -1 & 1 & k\end{vmatrix}\neq 0$，即 $(k+1)^2(k-2)\neq 0$。

k的取值範圍是 $\{k: k\neq -1$ 及 $k\neq 2\}$。

(ii) 設 $k=2$，從 (E) 得 $\begin{cases} 2x+\ \ y-z\ =p \\ \ \ x+2y+\ z\ =q \\ -x+\ \ y+2z=r \end{cases}$。

從以上的第一及第二條方程得 $x-y-2z\ =p-q$，把此方程與第三條方程比較，得 $p-q=-r$，即 $p-q+r=0$。

解 $\begin{cases} 2x+\ \ y\ -z\ =5 \\ \ x+2y+\ z\ =1 \end{cases}$，得 $x=3+t$，$y=-1-t$，$z=t$，其中 t 為任意實數。

Suggested Answer:

1. (a) (i) By direct calculation, $|PD| = \sqrt{|PA|^2 + |AD|^2} = \sqrt{5}$,

$|BD| = \sqrt{|BA|^2 + |AD|^2} = \sqrt{5}$ and $|PB| = \sqrt{|PA|^2 + |AB|^2} = \sqrt{8}$.

So, ΔPBD is an isosceles triangle. With PB as the base, its height is

$\sqrt{|PD|^2 - (\frac{1}{2}|PB|)^2} = \sqrt{3}$. Hence, the area of ΔPBD is $\sqrt{6}$.

(ii) The volume of $P - ABD$ is $\frac{1}{3}|PA|(\text{area of } \Delta ABD) = \frac{2}{3}$.

Let d be the distance from point A to plane PBD. Then the volume of $A - PBD$

is $\frac{d\sqrt{6}}{3}$. From $\frac{d\sqrt{6}}{3} = \frac{2}{3}$, we get $d = \sqrt{\frac{2}{3}}$.

(b) (i) From $PA \perp ABCD$, we get $DA \perp AP$. Together with $DA \perp AB$, we have

$AD \perp PAB$. As AD and BC are parallel, we get $BC \perp PAB$.

(ii) From the proof of (b)(i), we know that $AD \perp PAB$. So, $AD \perp AN$. …..(1)

Since N and M are the midpoints of PB and PC, respectively, we have $NM \parallel BC$.

Together with $AD \parallel BC$, we get $NM \parallel AD$. Hence, points A 、D 、M and N are

co-planar. Moreover, from (1), we have $NM \perp AN$. …..(2)

Since N and M are the midpoints of PB and PC, respectively, we have

$|MN| = \frac{1}{2}|BC| = 1 = |AD|$. …..(3)

Combining (1) - (3) above, $ADMN$ is a rectangle. So, $DM \parallel AN$. Hence,

DM is parallel to the plane PAB.

2. (a) (i) $f'(x) = 6x^2 - 18x + 12$, $f''(x) = 12x - 18$.

(ii) $f'(x) = 0 \Leftrightarrow x = 1$ or $x = 2$.

When $x < 1$, $f'(x) > 0$ and so $f(x)$ is increasing.

When $1 < x < 2$, $f'(x) < 0$ and so $f(x)$ is decreasing.

When $2 < x$, $f'(x) > 0$ and so $f(x)$ is increasing.

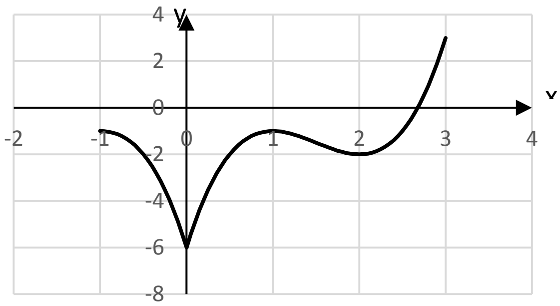
Hence, $f(1) = 0$ is a local maximum value, $f(2) = -1$ is a local minimum value.

$f''(x) = 0 \Leftrightarrow x = \frac{3}{2}$. When $x < \frac{3}{2}$, $f''(x) < 0$; when $\frac{3}{2} < x$, $f''(x) > 0$.

Hence, the inflection point of the curve $y = f(x)$ is $(\frac{3}{2}, -\frac{1}{2})$.

(iii)

(iv)



(b) Solving $\begin{cases} y = -x^2 + 3x \\ y = 2x^3 - x^2 - 5x \end{cases}$, we obtain $x = -2$ or $x = 0$ or $x = 2$.

When $-2 < x < 0$, the curve $y = -x^2 + 3x$ is below the curve $y = 2x^3 - x^2 - 5x$.

When $0 < x < 2$, the curve $y = -x^2 + 3x$ is above the curve $y = 2x^3 - x^2 - 5x$.

Hence, the required area is

$$\int_{-2}^{0} 2x^3 - x^2 - 5x - (-x^2 + 3x)\,dx + \int_{0}^{2} -x^2 + 3x - (2x^3 - x^2 - 5x)dx$$

$$= \int_{-2}^{0} 2x^3 - 8x\,dx + \int_{0}^{2} 8x - 2x^3\,dx$$

$$= \left[\frac{x^4}{2} - 4x^2\right]_{-2}^{0} + \left[4x^2 - \frac{x^4}{2}\right]_{0}^{2}$$

$$= 8 + 8$$

$$= 16.$$

3. (a) (i) From $\begin{cases} y = mx + c \\ x^2 = 8y \end{cases}$, we get $x^2 - 8mx - 8c = 0$. …… (1)

At the tangent point of L and P, (1) has a double root, its discriminant is 0.

Hence $c = -2m^2$.

(b) (i) From (a), suppose an equation for L_i is $y_i = m_i x - 2m_i^2$, $i = 1, 2$. As $A(h,k)$ is the intersection point of L_1 and L_2, from $\begin{cases} m_1 h - k = 2m_1^2 \\ m_2 h - k = 2m_2^2 \end{cases}$, we get $(h,k) = (2(m_1 + m_2), 2m_1 m_2)$.

(ii) Since L_1 and L_2 are perpendicular, $m_1 m_2 = -1$ and so $k = 2m_1 m_2 = -2$.

From $m_2 = -\frac{1}{m_1}$, we get $h = 2(m_1 - \frac{1}{m_1})$.

For any real number h, consider the equation $2m^2 - mh - 2 = 0$ where m is the unknown. Its discriminant $h^2 + 16 > 0$, and so it has real solutions. Thus, we know that $h = 2(m_1 - \frac{1}{m_1})$ can achieve any real value. Hence, the locus of point A is the straight line $y = -2$.

(iii) As the angle between L_1 and L_2 is $\frac{\pi}{4}$, we get $1 = \tan\frac{\pi}{4} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

When $m_1 = 2$, we have $\left|\frac{2 - m_2}{1 + 2m_2}\right| = 1$, that is, $3m_2^2 + 8m_2 - 3 = 0$. Solving, we get $m_2 = -3$ or $m_2 = \frac{1}{3}$. Hence, point A is $(-2, -12)$ or $(\frac{14}{3}, \frac{4}{3})$.

4. (a) From

$$\sin A = \sin(\frac{A+B}{2} + \frac{A-B}{2}) = \sin\frac{A+B}{2}\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

and

$$\sin B = \sin(\frac{B+A}{2} + \frac{B-A}{2}) = \sin\frac{B+A}{2}\cos\frac{B-A}{2} + \cos\frac{B+A}{2}\sin\frac{B-A}{2}$$

$$= \sin\frac{A+B}{2}\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\sin\frac{A-B}{2},$$

we get $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$.

(b)

$$
\begin{aligned}
& \sin A + \sin B + \sin C \\
= {} & 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C \\
= {} & 2\sin\frac{\pi-C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
= {} & 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
= {} & 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\frac{C}{2}\right) \\
= {} & 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\frac{\pi-(A+B)}{2}\right) \\
= {} & 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right) \\
= {} & 2\cos\frac{C}{2}\left(\cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2} + \left(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\right)\right) \\
= {} & 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}
\end{aligned}
$$

(c)(i) Let $P(n)$ be the proposition

“ $2\sin x\,[\cos x + \cos 3x + \cdots + \cos(2n-1)x] = \sin 2nx$ ”.

When $n = 1$, since $2\sin x\cos x = \sin 2x$, $P(1)$ is true.

Suppose $P(k)$ is true for some positive integer k. When $n = k + 1$,

$$
\begin{aligned}
& 2\sin x\,[\cos x + \cos 3x + \cdots + \cos(2(k+1)-1)x] \\
= {} & \sin 2kx + 2\sin x\cos(2k+1)x \\
= {} & \sin 2kx + 2\sin\frac{(2k+2)x-2kx}{2}\cos\frac{(2k+2)x+2kx}{2} \\
= {} & \sin 2kx + \sin(2k+2)x + \sin(-2kx) \\
= {} & \sin 2(k+1)x.
\end{aligned}
$$

Hence $P(k+1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for any positive integer n.

(ii) Obviously, when $x = 0, \pi, 2\pi$， $\sin x = 0$ and $\cos x + \cos 3x + \cos 5x \neq 0$.

From (i), when $0 \le x \le 2\pi$, $\cos x + \cos 3x + \cos 5x = 0$ if and only if $\sin 6x = 0$ and $x \neq 0, \pi, 2\pi$. Solving $\sin 6x = 0$, we get $x = \frac{n\pi}{6}$, $n = 1,2,3,4,5,7,8,9,10,11$.

5. (a)

$$\begin{vmatrix} a & a^2+1 & bc \\ b & b^2+1 & ac \\ c & c^2+1 & ab \end{vmatrix} = \begin{vmatrix} a & a^2+1 & bc \\ b-a & b^2-a^2 & c(a-b) \\ c-a & c^2-a^2 & b(a-c) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} a & a^2+1 & bc \\ 1 & b+a & -c \\ 1 & c+a & -b \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} a & a^2+1 & bc \\ 1 & b+a & -c \\ 0 & c-b & c-b \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix} a & a^2+1 & bc \\ 1 & b+a & -c \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix} a & a^2+1-bc & bc \\ 1 & b+a+c & -c \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)[a(a+b+c)-(a^2+1-bc)]$$

$$= (a-b)(b-c)(c-a)(ab+bc+ac-1)$$

(b) (i) The condition that (E) has a unique solution is $\begin{vmatrix} k & 1 & -1 \\ 1 & k & 1 \\ -1 & 1 & k \end{vmatrix} \neq 0$, that is,

$(k+1)^2(k-2) \neq 0$. The range of k is $\{k: k \neq -1$ and $k \neq 2\}$.

(ii) Let $k = 2$. Then, (E) becomes $\begin{cases} 2x + y - z = p \\ x + 2y + z = q \\ -x + y + 2z = r \end{cases}$.

From the first and second equations above, we get $x - y - 2z = p - q$.

Comparing this with the third equation, we get $p - q = -r$. So, $p - q + r = 0$.

Solving $\begin{cases} 2x + y - z = 5 \\ x + 2y + z = 1 \end{cases}$, we get $x = 3 + t$, $y = -1 - t$, $z = t$, where t is any real number.